Intertemporal Mean-Variance Efficiency with a Markovian State Price Density*

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Abstract

This paper extends Merton's continuous time (instantaneous) mean-variance analysis and the mutual fund separation theory. Given the existence of a Markovian state price density process, the optimal portfolios from concave utility maximization are instantaneously mean-variance efficient independent of the concave utility function's form. The Capital Asset Pricing Model holds with the market portfolio induced by the growth optimal portfolio. The Markowitz-Tobin mutual fund separation is extended to include the lognormal assumption for asset prices as a special case. Closed form solutions to the expected utility maximization of terminal portfolio value are derived. We present an example in which the state price processes are specified as a multivariate geometric Brownian motion and the asset prices follow a multivariate diffusion process with relatively general parameters.

JEL classification: C61; D92; G11; G12
Keywords: Mean-Variance Efficiency; Mutual Fund Separation; Utility Maximization; Equilibrium; Intertemporal CAPM; Markovian State Price Density

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Abstract

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1 Introduction

Alternative approaches have been sought for the mean-variance criterion to apply in dynamic investment analysis. Merton (1973) discusses the approach of applying the Markowitz (1952) static model instantaneously in time, i.e., minimizing the instantaneous standard deviation for a given target instantaneous mean rate of return . He developed the Intertemporal Capital Asset Pricing Model (ICAPM) and the mutual fund separation theory using stochastic control methodology. He showed that utility maximization is instantaneously mean-variance efficient and the continuous time analog of the static Capital Asset Pricing Model developed by Sharpe (1964) and Lintner (1965) holds when asset prices are log-normal. However, the log-normal assumption is only a sufficient condition. This paper extends the theory to a broader dynamic setting where the asset price behavior mechanism is determined by a Markovian state price density process.

State prices are used to characterize the security prices–one for each state of the world. The price of a given security is simply a linear combination of its payoffs in all states with state prices as the coefficients. One can interpret the state price as the marginal cost for obtaining one unit of the asset in that state. The existence of a state price process is closely related to the concept of arbitrage free prices. Harrison and Kreps (1979) derived the existence of a state price process if the market is arbitrage free. The main characterization of the state price process is that all securities prices or portfolio values are discounted martingles under the risk neutral probability induced by the state price process.

Since state price process plays an important role in our discussions, we need to elaborate further. Conditions for the existence of a state price process are satisfied if the market is arbitrage free; see Harrison and Kreps (1979) and Harrison and Pliska (1981). If the market is complete, the state price process is unique almost surely. By definition, state prices are determined by future uncertainty and the payoff for each state of the world. There is no strong economic reason that the state prices should be path dependent. Knowing the price (or marginal cost) for the current state, the future state prices should be solely determined by economic uncertainty. The current state of research focuses on finding the implication about the state prices from individual stocks on which the whole market risk and return tradeoff is established. Hence, it is a proper way to assume the existence of such a state price process that newly entering stocks or securities must be jointly priced by the absence of arbitrage (including equilibrium settings) and by its payoff structure. The reason is that the economy is dynamically stable so that all securities (currently existing or newly entering) are priced efficiently by the state price process.

As shown in Duffie (2000), the state price beta model has a similar format to the static CAPM, but little research in an equilibrium setting has been presented in the literature. That is, it is not clear whether the risky fund in the state price beta model is closely related to the market portfolio. With the setting of a Markovian state price process, we can affirm that the intertemporal CAPM in Merton's formulation holds with the market portfolio being induced by the growth optimal portfolio. We prove that all portfolios maximizing expected utility of terminal wealth (for simplicity, we omit consumption) are on the instantaneous mean-variance efficient frontier. Hence, utility maximization is equivalent to instantaneous mean-variance analysis in this case.

Mutual fund separation theory concerns alternative approaches for allocating wealth to primary assets. An immediate advantage of the theory is that investment decisions can be divided into holdings in a few risky mutual funds which issue shares of their own for purchase by individual investors. The separation is executable because mutual fund managers are instructed to hold the proportions of the individual assets independent of investors' preferences and their wealth distributions. In the static CAPM setting, if asset returns are normally distributed or the investors' utilities are quadratic, all investors can alternatively invest in two mutual funds that are constructed using the primary assets. Merton (1973) derived a two mutual fund separation for fixed investment opportunities (constant μ_t and σ_t and a three mutual fund separation for stochastic interest rate (stochastic α_t but fixed μ_t and σ_t). We derive a two mutual fund separation that includes the existing theory as a special case. Since lognormal prices imply a lognormal state price process, which is a Markovian process, the setting of lognormal prices is a special case of the Markovian state prices. The growth optimal portfolio (maximizing expected log of wealth) is the risky fund in the two fund separation. We show that the equilibrium market portfolio is induced by the growth optimal portfolio. All investors are indifferent between investing in the two mutual funds, the riskless security and the growth optimal portfolio, and investing in the combination of market primary assets.

The optimal portfolio policies can be identified as a function of the state price and time variable by solving the associated partial differential equations. The solution techniques follow those developed in Harrison and Kreps (1979) and Cox and Huang (1989). We present a closed form solution to the optimal portfolio policy which only depends on the growth optimal portfolio and individual risk aversion.

2 Portfolio Dynamics and the Beta Model

2.1 Model Setting

Assume a complete probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $[0, T]$, where T is a strictly positive real number. Let $z_t = (z_{1t}, \cdots, z_{Nt})^\top$ denote an n-dimensional standard Brownian motion which generates a filtration $\mathbf{F} = \{\mathcal{F}_t \subseteq \mathcal{F}; t \in [0, T]\}$, which describes the economic uncertainty. A stochastic process W_t is called adapted to $\mathbf{F}$ if W_t is measurable with respect to $\mathcal{F}_t$.

The financial market is absent of arbitrage in the sense that no trading strategy can make a "free lunch", which implies the existence of a state price process. This setting has been standard for pricing a security by arbitrage. We assume the state price density process is a Markovian diffusion process

$$\frac{d\xi_t}{\xi_t} = \alpha(t, \xi_t)\ dt + \beta(t, \xi_t)^\top\ dz_t, \qquad \xi_0 = 1, \tag{1}$$

where $\alpha(t, \xi)$ and $\beta(t, \xi)$ are at most functions of t and ξ and satisfy the Novikov condition (Karatzas and Shreve 1990) for the purpose of stochastic integrability.

The assumption of the Markovian property for the state price density process ξ_t is credible. First, ξ_t is the marginal cost of increasing one unit of payoff in some state, hence, it might be independent of the past information about the economic uncertainty. The current state price is sufficient "statistics" for determining the marginal cost. Secondly, this model contains the case of log-normal asset prices. State prices are lognormally distributed if asset prices jointly follow a multivariate geometric Brownian motion. Also, many asset pricing models such as Black and Scholes (1973) option pricing model are based on the assumption of lognormality for prices. Thus, it is of interest to extend the related financial theoretical model to include a broader setting for asset prices.

We use α_t and β_t for short whenever there is no confusion. We further assume no transaction costs, no liquidity constraints, and the state price density process ξ_t satisfies the following two assumptions.

Assumption 1. *For every asset price (portfolio) process W_t, $\xi_t W_t$ is a martingle terminated at $\xi_T W_T$.*

Assumption 2. *If $\xi_t W_t$ is a martingale, then W_t is a valid portfolio process, i.e., it can be replicated using market primary assets.*

The martingale assumption with the positivity of the state price process ξ_t rules out the arbitrage opportunities. So, ξ_t is restricted to a positive Markovian process. Following Harrison and Kreps (1979) and Harrison and Pliska (1981), Assumption 1 implies that the expected value of a security or a portfolio should be the same as its current value after being adjusted by the state prices over time. Assumption 2 is a similar hypothesis to market completeness, but it is a weaker version than that in the literature. We only impose that there is a state price density process with the characterization shown as Equation (1) and Assumptions 1 and 2.

2.2 The Dynamics of the Portfolio Value

Let W_t be a portfolio value (asset price) process. By Assumption 2, $\xi_t W_t$ is a martingale. Hence, by the martingale representation theorem, one can derive

Proposition 1. *For any asset (portfolio) in the market, there exists a predictable stochastic process ϕ_t such that the asset price (portfolio value) process W_t follows the stochastic differential equation*

$$\frac{dW_t}{W_t} = \left[\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t\right] dt + \left[\phi_t^\top - \beta_t^\top\right] \; dz_t. \tag{2}$$

Proof. See Appendix A.

Proposition 1 characterizes the dynamics of the asset prices by the state price density process. There are advantages of modelling the stock price dynamics as such an equilibrium representation for a given state price density process. The state prices are implied by the overall market performance but not by individual stocks. Equilibrium prices of individual stocks should follow Equation (2) to ensure that the market is arbitrage free. A criterion for admissible portfolios is

Proposition 2. *If the stochastic process*

$$\frac{dW_t}{W_t} = \mu_t \; dt + \sigma_t^\top \; dz_t. \tag{3}$$

represents an admissible portfolio process, then μ_t and σ_t satisfy the linear equation

$$\alpha_t + \mu_t + \sigma_t^\top \beta_t = 0. \tag{4}$$

Conversely, if (4) *holds and σ_t satisfies the Novikov condition of integrability (Karatzas and Shreve 1990), then Equation* (3) *represents an admissible portfolio process.*

Proof. See Appendix A.

Equation (4) determines how the instantaneous mean μ_t and the volatility vector σ_t of an asset return are related. Some special cases are of interest. If $\phi_t = \beta_t$ with probability one, then Equation (2) implies that

$$\frac{dW_t}{W_t} = -\alpha_t \; dt$$

which represents the riskless asset. Hence, the short term interest rate is $-\alpha_t$, where α_t should be negative by the definition of state price density. This equation and Assumptions 1 and 2 imply that there exists a riskless asset. We either assume the existence of a (local) risk free asset or that it can be generated with the market existing securities. If $\phi_t = 0$, then Equation (2) implies that

$$\frac{dW_t}{W_t} = (\beta_t^\top \beta_t - \alpha_t) \; dt - \beta_t^\top \; dz_t$$

which represents the rate of return on the growth optimal portfolio, i.e., the process ξ_t^{-1} which is equivalent to expected log utility maximization; see Section 3 for the justification of this.

Equation (2) expresses the asset price or the portfolio value as a general diffusion process whose coefficients may not only depend on the state prices but also may depend on the current asset price or portfolio value. Generally speaking, asset prices or portfolio values are adapted to the information filtration $\mathcal{F}_t$ due to the choice of ϕ_t. Hence, the asset prices are allowed to be general diffusion process without many restrictions.

2.3 The State Price Beta Model

Let $\mu_{\xi t} = \beta_t \beta_t^\top - \alpha_t$ and $\sigma_{\xi t}^\top \sigma_{\xi t} = \beta_t \beta_t^\top$ denote the mean and variance of the instantaneous rate of return of the growth optimal portfolio. By Equation (4), the following *state price beta model* must hold for any asset price or portfolio value W_t with instantaneous mean return μ_t and instantaneous volatility vector σ_t

$$\mu_t + \alpha_t = \frac{\sigma_t^\top \beta_t}{\sigma_{\xi t}^\top \beta_t} \, (\mu_{\xi t} + \alpha_t), \tag{5}$$

where $-\sigma_t^\top \beta_t$ is the covariance of the rates of returns on W_t and the growth optimal portfolio ξ_t^{-1} and $-\sigma_{\xi t}^\top \beta_t = \beta_t^\top \beta_t$ is the variance of the rate of return on ξ_t^{-1}. Does Equation (5) look like the static CAPM equation? Since the growth optimal portfolio consists of the investment in the riskless asset and its asset weighting is different, it is not the market portfolio. However, the market portfolio is closely related to the growth optimal portfolio in our setting. The weights of the market portfolio in all risky assets are proportional to those of the growth optimal portfolio. We defer the justification of this theory to the next section when we discuss mutual fund separation.

3 Mean-Variance Efficiency and Mutual Fund Separation

Merton (1973) developed the continuous time analog of static mean-variance analysis. He concluded that, if the asset prices jointly follow a multivariate lognormal process, the growth optimal portfolio is instantaneous mean-variance efficient and two fund separation holds. Ross (1978b) developed conditions for static portfolio separation; see also Ross (1977) for restrictions on short sales and Ziemba (1974) for stable distribution separation. We now generalize.

The instantaneous mean-variance analysis is an optimization model that minimizes the instantaneous variance constrained with a desired instantaneous rate of return μ_t. From Equation (2), the optimization model is

$$\begin{aligned} \min_{\phi_t} \quad & (\phi_t - \beta_t)^\top (\phi_t - \beta_t) \\ s.t. \quad & \beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t = \mu_t \end{aligned} \tag{6}$$

where ϕ_t is the predictable process that is determined by admissible portfolio strategies. The implementation of the portfolio strategy is discussed later in this section.

Theorem 1. *The instantaneous mean-variance efficient frontier is a straight line in mean and standard deviation space with a slope of $\sqrt{\beta_t^\top \beta_t}$, i.e., the optimal mean rate of return μ_t and its standard deviation $\sqrt{\sigma_t^\top \sigma_t}$ are related as*

$$\frac{\mu_t + \alpha_t}{\sqrt{\sigma_t^\top \sigma_t}} = \sqrt{\beta_t^\top \beta_t}, \quad \mu_t > -\alpha_t. \tag{7}$$

Proof. See Appendix A.

Since $-\alpha_t$ is the rate of return on the riskless asset, Equation (7) is the continuous time version of Markowitz (1952) static model. In the dynamic setting, the instantaneous Sharpe ratio is the instantaneous standard deviation of the state prices. At any point in time, the Sharpe ratio of any efficient portfolio is constant, independent of individual asset movements but equal to the instantaneous standard deviation of the state price. Equation (7) also shows that the intertemporal efficient frontier is a ray in mean and standard deviation space, see Figure 1.

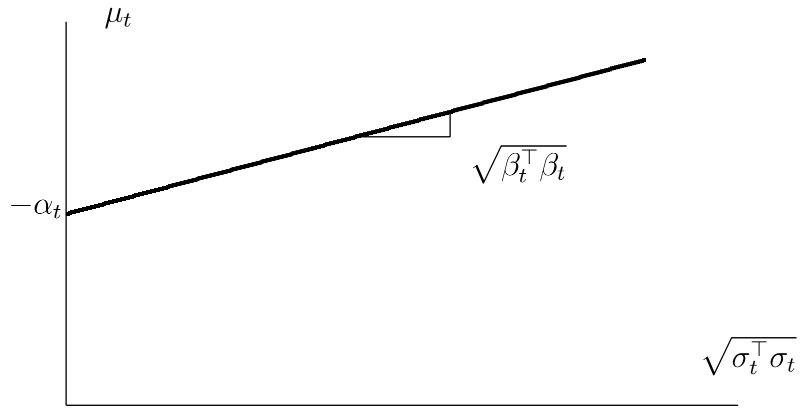

Figure 1: The Instantaneous Efficient Frontier at Time t

We now derive the conditions for a portfolio to be mean-variance efficient. Equations (4) and (7) imply that

$$-\sigma_t^\top \beta_t = \sqrt{\sigma_t^\top \sigma_t}\sqrt{\beta_t^\top \beta_t}. \tag{8}$$

If Equation (8) holds, then the portfolio is instantaneously mean-variance efficient. From Equation (4)

$$\mu_t + \alpha_t = -\sigma_t^\top \beta_t = \sqrt{\sigma_t^\top \sigma_t}\sqrt{\beta_t^\top \beta_t},$$

which implies that Equation (7) holds. Therefore, the growth optimal portfolio ξ_t^{-1} (with $\sigma_t = -\beta_t$) is on the efficient frontier. We now derive the conditions for an arbitrary portfolio to be instantaneous mean-variance efficient.

Proposition 3. *A portfolio is instantaneous mean-variance efficient if and only if*

$$\sigma_t = -a_t\ \beta_t,$$

where a_t is a positive scalar.

Proof. See Appendix A.

Mutual fund separation theory concerns alternative approaches for allocating wealth to primary assets. An advantage is that investment decisions can be divided into two parts by the establishment of two financial intermediaries (mutual funds) to hold all individual assets and to issue shares of their own for purchase by individual investors. The separation is executable because mutual fund managers are instructed to hold the proportions of the individual assets independent of investors' preferences and wealth distribution. If asset returns are normally distributed or the investor's utility is quadratic, all investors can alternatively invest in two mutual funds that are constructed using the primary assets. Merton (1973) derived a two mutual fund separation for fixed investment opportunities (constant μ_t and σ_t and a three mutual fund separation for stochastic interest rate (stochastic α_t but fixed μ_t and σ_t). Rudolf and Ziemba (2003) extended this model to include liabilities as a fourth mutual fund. We extend this result to the general case when the state price density process is Markovian.

Let $U(x)$ be a smooth concave utility function[1]. Levy and Markowitz (1979) studied the approximation of expected utility maximization by a function of mean and variance and Kallberg and Ziemba (1983) by risk aversion measures. Pliska (1986) studied the optimal portfolio rules in a stochastic calculus model. Assume investors' decisions are based on maximizing the expected utility of the terminal wealth. The optimization model is

$$\begin{aligned} \max_{W_T} \quad & E\left[U(W_T)\right] \\ s.t. \quad & E\left[\xi_T W_T\right] = W_0. \end{aligned} \tag{9}$$

The constraint in (9) is due to Assumption 1.

Having determined the mean-variance efficiency of a portfolio, we now state and prove our main result.

[1]To simplify the mathematical derivation, the utility function is required to be smooth and concave so that $U(x)$ is twice differentiable and its first order derivative is invertible. However, the non-satiation is relaxed to include a broader range of utility function forms. For example, the quadratic utility function is included in our discussion.

Theorem 2. *Assume that the state price density process is Markovian. Then*

i) *all investment portfolios that maximize the expected utility of terminal wealth are instantaneous mean-variance efficient; and*

ii) *two fund separation applies. The growth optimal portfolio can be chosen as the risky fund. Let $F(t, \xi_t)$[2] be the optimal wealth at time t and the proportion invested in the growth optimal portfolio $\theta_{\xi t}$. Then*

$$\theta_{\xi t} = -\frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}, \tag{10}$$

where $\frac{\partial F}{\partial \xi}$ is the partial derivative of $F(t, \xi)$ with respect to ξ.

Proof. See Appendix A.

Theorem 2 indicates that investors are indifferent between investing in two mutual funds and investing in a combination of market tradable assets. The two funds can be chosen as the riskless asset and the growth optimal portfolio. However, the optimal portfolio weights are functions of calender time and the value of the risky fund. There are no incentives for investors to know the performance of the individual stocks, but managers of the risky fund fully take the responsibility of managing the fund. This theory is consistent with the operation of financial investment in the market where mutual funds are acquired by individual investors.

In Section 2, we have asserted that the market portfolio is closely related to the growth optimal portfolio without providing a suitable argument. This can now be seen in view of Theorem 2. The market portfolio describes the allocation of the total market capitalization which does not include the riskless asset. Mathematically the market portfolio is a time

[2] Later in Appendix A, it is shown that the optimal portfolio value of maximizing expected utility is a function of time and the value of the state prices.

varying vector at any point in time. Since each investor only invests in two mutual funds, the riskless asset and the growth optimal portfolio, the market portfolio must have the same ratios among all risky assets as the growth optimal portfolio does in equilibrium. We state this theory as

Theorem 3. *Denote the weights of the growth optimal portfolio by $x_{\xi t}$ and let $m_t = x_{\xi t}^{\top}\iota_M$. The weight vector of the market portfolio, x_{mt}, is*

$$x_{mt} = \frac{1}{m_t} x_{\xi t} \quad (m \neq 0).$$

Proof. See Appendix A.

The static CAPM is an equilibrium model. In the static mean-variance analysis, all investors hold the riskless asset and the market portfolio. This is also true in the continuous time framework. By Theorem 3, the optimal weights of the market portfolio and the growth optimal portfolio have the same ratio among all risky assets at any point in time. Hence, all investors hold the market portfolio and the riskless asset. To have a complete continuous time analog of the static CAPM, we need to prove that the market portfolio is instantaneously mean-variance efficient and all investors hold the risk free asset and the market portfolio. Corollary 1 confirms this.

Corollary 1. *Given a Markovian State price density, the market portfolio is instantaneously mean-variance efficient and the Intertemporal Capital Asset Pricing Model holds .*

Proof. See Appendix A.

Hence, all investment strategies are viable through investing in the two mutual funds. Interestingly, it is implied that all optimal portfolios have the same ratios among all risky assets, though the weights in the risky assets might be different, depending on the level of risk aversion. These ratios are exactly the same as those from the logarithmic utility maximization. This result is consistent with the study of Rubinstein (1976), which presents a case that the logarithmic utility model is taken as the premier model of financial markets.

So far, we have not discussed how to implement an optimal portfolio strategy, given that we derived its structure. We now discuss this. Based on Theorems 2 and 3, we need only to find the optimal portfolio rules for the growth optimal portfolio. The solution to the growth optimal portfolio can be found in textbooks, such as Merton (1992), that deal with continuous time investment/consumption problems. Assume there are $M + 1$ securities for investment: A riskless asset B with instantaneous rate $-\alpha_t$ and M risky assets with price vector $S = (S_1, \ldots, S_M)$ which follows the stochastic equation

$$\frac{dS_t}{S_t} = \Gamma_t \; dt + \Sigma_t \; dz,$$

where Γ_t is an $M \times 1$ vector of instantaneous mean returns and $\Sigma_t \Sigma_t^\top$ is an $M \times M$ matrix of the instantaneous variance-covariance at time t. For any portfolio rule x_t in the risky assets, the corresponding portfolio value W_t is determined by

$$\begin{aligned} \frac{dW_t}{W_t} &= (1 - x_t^\top \mathbf{1}) \frac{dB}{B} + x_t^\top \frac{dS}{S} \\ &= [-\alpha_t + x_t^\top (\Gamma_t + \alpha_t \mathbf{1})] dt + x_t^\top \Sigma_t \; dz. \end{aligned}$$

Hence, comparing with the dynamics of the optimal growth portfolio value yields the vector of the weights for the growth optimal portfolio

$$x_{\xi t} = -(\Sigma_t \Sigma_t^\top)^{-1} (\Sigma_t \beta_t) \tag{11}$$

assuming that Σ_t is of full row rank, i.e., there are at least as many Brownian motions as linearly independent primary assets. Equation (11) can be viewed as the ratio of the covariances of the risky assets and the market portfolio (as shown in Appendix A, $-\beta_t$ is the volatility vector for the market portfolio) and the variances of the risky assets.

Having determined the optimal portfolio weights of the growth optimum, we now turn to the calculation of the optimal portfolio policy of an arbitrary smooth concave utility maximization. Since all utility maximization portfolios are instantaneous mean-variance efficient, we need only to find the optimal proportion that is invested in the growth optimal portfolio and the risk free asset. Suppose the investor's desired return is μ_t (which basically depends on the investor's risk aversion). Let λ_t be the portfolio weight invested in the growth optimal portfolio, then the value of such a portfolio is

$$\frac{dP}{P} = (1-\lambda_t)\frac{dB}{B} + \lambda_t \frac{d\xi^{-1}}{\xi^{-1}}.$$

Setting the instantaneous rate equal to μ_t yields the optimal λ_t

$$\lambda_{opt} = (\beta_t^\top \beta_t)^{-1}(\mu_t + \alpha_t). \tag{12}$$

Therefore, the optimal portfolio weights in the risky assets are

$$x_{\mu t} = \lambda_{opt} \cdot x_{\xi t} = -(\beta_t^\top \beta_t)^{-1}(\mu_t + \alpha_t)(\Sigma_t \Sigma_t^\top)^{-1}(\Sigma_t \beta_t)$$

and the weight in the riskless asset is

$$1 - x_{\xi t}^\top \iota_M = 1 + (\beta_t^\top \beta_t)^{-1}(\rho + \alpha_t) \cdot [\iota_M^\top (\Sigma_t \Sigma_t^\top)^{-1}(\Sigma_t \beta_t)]$$

where ι_M is the vector of 1's of length M.

4 A Special Case

To illustrate the model and the theoretical results derived, we present a relatively general case where the state price process follows the multivariate geometric Brownian motion

$$\frac{d\xi}{\xi} = \alpha dt + \beta^\top dz$$

where α and β are constant[3]. Suppose there are $M+1$ assets for investment. The rate of return on the riskless asset must equal $-\alpha$. Let $\Gamma_t^{(i)}$ and $\Sigma_t^{(i)}$ be the ith asset's instantaneous

[3]Generally, α and β can be time varying and dependent on the state price level. For simplicity, our discussion is restricted to the case of constant parameters.

rate of return and the instantaneous volatility vector, respectively. Then, the asset prices follow the stochastic differential equations

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \Gamma_t^{(i)} dt + \Sigma_t^{(i)\top} dz_t, \quad \forall i = 1, \ldots, M.$$

By Equation (4), Γ_i and Σ_i must satisfy

$$\Gamma_t^{(i)} + \alpha + \Sigma_t^{(i)\top} \beta = 0, \quad \forall i = 1, 2, \cdots, M$$

which implies that

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (-\alpha - \Sigma_t^{(i)\top} \beta) dt + \Sigma_t^{(i)\top} dz_t, \quad \forall i = 1, \ldots, M.$$

The $\Sigma_t^{(i)}$ may be specified under minor regularity conditions. This provides us with choices for specifying the asset price model assuming Markovian state prices. If each $\Sigma_t^{(i)}$ is a constant vector, then the asset price model reduces to the popular lognormal model for prices. If each $\Sigma_t^{(i)}$ is proportional to the price level, this model reduces to some affine models.

Denote $S_t = [S_t^{(1)}, \ldots, S_t^{(M)}]^\top$ and $\Sigma_t = [\Sigma_t^{(1)}, \cdots, \Sigma_t^{(M)}]^\top$. Then

$$\frac{dS_t}{S_t} = (-\alpha \iota_M - \Sigma_t \beta) dt + \Sigma_t dz_t$$

where Σ_t is an $M \times N$ matrix and ι_M the column vector of 1's of length M. To avoid non-singularity, assume Σ_t is of full row rank.

It was shown in Section 3 that utility maximization is equivalent to a mean-variance criterion. The vector of growth optimal portfolio weights in the M risky assets is, by (11)

$$x_{\xi t} = -(\Sigma_t \Sigma_t^\top)^{-1} (\Sigma_t \beta). \tag{13}$$

Hence by (12), with μ_t as the desired level of expected return, the vector of the optimal portfolio weights in the M risky assets is

$$x_{\mu t} = -(\beta^\top \beta)^{-1} (\mu_t + \alpha)(\Sigma_t \Sigma_t^\top)^{-1} (\Sigma \beta) \tag{14}$$

and the weight invested in the riskless asset is

$$1 - \iota_M^\top x_{\mu t}$$

where ι_M is the vector of 1's of length M.

5 Concluding Remarks

With the existence of a Markovian state price density process, we proved that the intertemporal CAPM holds with the market portfolio induced by the growth optimal portfolio. Especially, all investment portfolios of the concave utility maximization of terminal wealth are instantaneously mean-variance efficient. This result extends the two fund separation theory to a broader setting of the market asset returns. The growth optimal portfolio can be chosen as the risky fund. The efficient frontier was solely determined by the state price density process.

The efficient frontier is uniquely determined by the mean and the standard deviation of the state price process. The efficient frontier is also a ray which intersects the vertical axis corresponding to the riskless rate with a slope equal to the instantaneous standard deviation of the state price.

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A Appendix

Proof of Proposition 1:

By Assumption 2, $\xi_t W_t$ is a martingale that terminates at $\xi_T W_T$. By the martingale representation theorem, see Øksendal (1998), there exists a unique adapted stochastic process ϕ_t such that

$$\frac{d\xi_t W_t}{\xi_t W_t} = \phi_t^\top \, dz_t.$$

Since the growth optimal portfolio follows the stochastic differential equation

$$\frac{d\xi_t^{-1}}{\xi_t^{-1}} = (\beta_t^{-1}\beta_t - \alpha_t) \, dt - \beta_t^\top \, dz_t.$$

By Itô's formula

$$\begin{aligned}
\frac{dW_t}{W_t} &= \frac{d\xi_t^{-1}}{\xi_t^{-1}} + \frac{d\xi_t W_t}{\xi_t W_t} + \frac{d\xi_t W_t}{\xi_t W_t} \cdot \frac{d\xi_t^{-1}}{\xi_t^{-1}} \\
&= (\beta_t^{-1}\beta_t - \alpha_t) \, dt - \beta_t^\top \, dz_t + \phi_t^\top \, dz_t - \phi_t^\top \beta_t \, dt \\
&= (\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t) \, dt + (\phi_t - \beta_t)^\top \, dz_t.
\end{aligned} \tag{15}$$

Proof of Proposition 2:

By Assumption 1 and Proposition 1, there is an adapted stochastic process ϕ_t such that

$$\mu_t = \beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t, \quad \sigma_t = \phi_t - \beta_t.$$

Cancelling out ϕ_t yields

$$\mu_t + \alpha_t + \sigma_t^\top \beta_t = 0.$$

Conversely, if the above equation hold, then $\xi_t W_t$ is a martingale. By Assumption 2, W_t is a portfolio process.

Proof of Theorem 1:

The Lagrangian of the optimization model (6) is

$$\mathcal{L}(\phi_t, \lambda) = \left[(\phi_t - \beta_t)^\top (\phi_t - \beta_t)\right] - \lambda \left(\beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t - \mu_t\right).$$

The first order conditions imply that

$$\begin{cases} 2(\phi_t - \beta_t) + \lambda \beta_t = 0 \\ \beta_t^\top \beta_t - \alpha_t - \phi_t^\top \beta_t - \mu_t = 0. \end{cases}$$

Multiplying the first equation by β_t and $\phi_t - \beta_t$ yields

$$\lambda = -\frac{2(\phi_t - \beta_t)^\top \beta_t}{\beta_t^\top \beta_t}$$
$$2(\phi_t - \beta_t)^\top (\phi_t - \beta_t) = -\lambda(\phi_t - \beta_t)^\top \beta_t.$$

The second equation above yields

$$(\phi_t - \beta_t)^\top \beta_t = -(\mu_t + \alpha_t).$$

Therefore

$$(\phi_t - \beta_t)^\top (\phi_t - \beta_t) = \frac{(\mu_t + \alpha_t)^2}{\beta_t^\top \beta_t},$$

where $\sigma_t = \phi - \beta_t$ is the volatility vector of the optimal portfolio.

Proof of Proposition 3:

Let W_t be a portfolio with $\sigma_t = -a_t\beta_t$, where a_t is a scalar. Thus, Equation (8) is satisfied. Hence, by Proposition 2, W_t is instantaneous mean-variance efficient. Conversely, if W_t is mean-variance efficient but $\sigma_t \neq -a_t\beta_t$ for any $a_t > 0$, then

$$(\sigma_t + a_t\beta_t)^\top (\sigma_t + a_t\beta_t) > 0,$$

which implies that

$$(\sigma_t^\top \beta_t)^2 - \sigma_t^\top \sigma_t \cdot \beta_t^\top \beta_t > 0.$$

This contradicts Theorem 1. Hence, there exists a positive scalar process a_t such that $\sigma_t = -a_t\beta_t$.

Proof of Theorem 2:

Let $U(x)$ be a smooth concave utility function so that the first order derivative is invertible. A utility maximizer solves the optimization model (9) with the Lagrange multiplier λ as

$$\mathcal{L}(W_T, \lambda) = E\left[U(W_T)\right] - \lambda\left(E\left[\xi_T W_T\right] - W_0\right).$$

The first order conditions are

$$\begin{cases} U_x(W_T) - \lambda\xi_T = 0 \\ E\left[\xi_T W_T\right] = W_0, \end{cases}$$

where $U_x(\cdot)$ is the first order derivative. Let $U_x^{-1}(\cdot)$ be its inverse function, then

$$W_T = U_x^{-1}(\lambda \xi_T)$$

where λ satisfies

$$E\left[\xi_T \cdot U_x^{-1}(\lambda \xi_T)\right] = W_0.$$

Let W_t be the portfolio value at time t. Since $\xi_t W_t$ is a martingale, then

$$\xi_t W_t = E\left[\xi_T U_x^{-1}(\lambda \xi_T)|\mathcal{F}_t\right].$$

Since ξ_t is Markovian, the wealth W_t must be a function of t and ξ_t, i.e., there exists a deterministic function $F(t, \xi)$ such that

$$W_t := F(t, \xi_t) = \xi_t^{-1}\ E\left[\xi_T \cdot U_x^{-1}(\lambda \xi_T)|\mathcal{F}_t\right].$$

By Itô's formula

$$\begin{aligned} \frac{dW_t}{W_t} &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial \xi}\ d\xi_t + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\ (d\xi_t)^2\right) \\ &= \frac{1}{W_t}\left(\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \frac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2}\right)\ dt + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\beta_t^\top dz_t. \end{aligned} \tag{16}$$

By Theorem 1, W_t is instantaneous mean-variance efficient. This proves the first part of Theorem 2.

Equation (4) implies

$$\frac{1}{W_t}\left(\frac{\partial F}{\partial t} + \alpha_t \xi_t \frac{\partial F}{\partial \xi} + \tfrac{1}{2}\xi_t^2 \beta_t^\top \beta_t \frac{\partial^2 F}{\partial \xi^2}\right) + \alpha_t + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\beta_t^\top \beta_t = 0.$$

Denote the riskless asset by B_t, i.e., $dB_t = -\alpha_t\ B_t dt$. Hence, one can derive from the above equation that

$$\frac{dW_t}{W_t} = \left(1 + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}\right) \cdot \frac{dB_t}{B_t} - \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi} \cdot \frac{d\xi_t^{-1}}{\xi_t^{-1}}. \tag{17}$$

This means that the optimal portfolio can be replicated by the riskless asset and the growth optimal portfolio. The optimal portfolio rule is to invest the proportion of $-\frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi}$ in the growth optimal portfolio and $(1 + \frac{\xi_t}{W_t}\frac{\partial F}{\partial \xi})$ in the riskless asset. Thus, the two fund separation theorem applies.

Proof of Theorem 3:

Suppose there are L investors with endowment wealth (in market share value) of $W_1, \cdots, W_L$. By Theorem 2, each investor holds a portfolio of the riskless asset and the growth optimal portfolio. Let $\gamma_1, \cdots, \gamma_L$ be the proportion invested in the growth optimal portfolio, correspondingly, and let $x_{\xi t}$ be the vector of growth optimal portfolio weights in the rsiky assets. Then, the total capitalization for each risky asset is

$$(\gamma_1 W_1 + \gamma_2 W_2 + \cdots + \gamma_L W_L) x_{\xi t},$$

and the total market capitalization equals

$$(\gamma_1 W_1 + \gamma_2 W_2 + \cdots + \gamma_L W_L) x_{\xi t}^\top \iota_M.$$

Hence, the vector of the market portfolio weights is

$$\begin{aligned} x_{mt} &= \frac{(\gamma_1 W_1 + \gamma_2 W_2 + \cdots + \gamma_L W_L) x_{\xi t}}{(\gamma_1 W_1 + \gamma_2 W_2 + \cdots + \gamma_L W_L) x_{\xi t}^\top \iota_M} \\ &= \frac{1}{m_t} x_{\xi t} \end{aligned}$$

where $m_t = x_{\xi t}^\top \iota_M$. It is assumed that $m_t \neq 0$, which is guaranteed as long as at least one risky asset has a nonzero correlation with the state price density.

Proof of Corollary 1:

From Theorem 3, the market portfolio is

$$x_{mt} = \frac{1}{m_t} x_{\xi t}.$$

Hence, the growth optimal portfolio can be represented as

$$\frac{d\xi_t^{-1}}{\xi_t^{-1}} = (1 - m_t)\frac{dB_t}{B_t} + m_t \frac{dM_t}{M_t}$$

where M_t stands for the level of the market portfolio and B_t is the riskless asset earning rate of return $-\alpha_t$. Therefore,

$$\begin{aligned} \frac{dM_t}{M_t} &= \frac{1}{m_t}\frac{d\xi_t^{-1}}{\xi_t^{-1}} - \frac{1 - m_t}{m_t}\frac{dB_t}{B_t} \\ &= \frac{1}{m_t}\left[(\beta_t^\top \beta_t - \alpha_t)dt - \beta_t dz\right] + \frac{1 - m_t}{m_t}\alpha_t dt \\ &= (\beta_t^\top \beta_t - \alpha_t)dt - \frac{1}{m_t}\beta_t^\top dz. \end{aligned}$$

By Proposition 3, the market portfolio is mean-variance efficient.